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Half-Range Completeness for the Fokker–Planck Equation with an External Force

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This paper deals with a generalization of a classical result obtained by R. Beals and V. Protopopescu for the Fokker–Planck equations to the case in which a constant external force is present.

KEY WORDS: Fokker–Planck equation; Brownian motion; half-range completeness; linear transport theory; indefinite Sturm–Liouville problems; kinetic layers; boundary value problems.

1. INTRODUCTION

Some years ago, Beals and Protopopescu[(1)] proved a half-range completeness theorem for the linear, stationary, one-dimensional Fokker–Planck equation (FPE). In fact, they proved that half of the eigenfunctions computed by Pagani[(3)] corresponding to a positive velocity argument is complete on the real positive half-axis (the same is true for negative arguments on the negative half-axis).

We generalize that result to the case in which the FPE also contains a term representing the action of a constant external force on the particles of the system described by the equation. Interest in this problem has grown in connection with the study of evaporation phenomena: Burschka and Titulaer[(7)] attacked the problem of determining the distribution function for a Brownian particle in the one-particle phase space in connection with the kinetic boundary layer solution of the FPE; refs. 5 and 6 examine the problem of the growth of small droplets, by using the Klein–Kramers equation. In a different context, ref. 13 studies the half-range expansion problem for the FPE. In refs. 8 and 9 the problem of calculating the first passage

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time for the Ornstein–Uhlenbeck process is studied, while Marshall and Watson obtained[2] the explicit analytic form of the corresponding eigenfunctions, without proving the completeness of the system, in order to obtain the analytic solution of some boundary layer problems. In assuming the completeness of the system of eigenfunctions, they say "The corresponding result for $\alpha = 0$ has been proved and we think it is only a matter of time to extend the proof to our case." We want to accomplish this task now.

The plan of the work is the following: in the next section the problem is introduced together with the proper setting of function spaces; then we recall some previous results on indefinite Sturm–Liouville problems, while in the final section we deal with the half-range completeness property of the corresponding eigenfunctions by applying a theorem proved by Beals.[4]

2. STATEMENT OF THE PROBLEM

We study the following equation:

$$v \frac{\partial \psi}{\partial x}(x, v) = \left(\gamma \frac{kT}{m} \frac{\partial^2}{\partial v^2} + \gamma \frac{\partial}{\partial v} v \right) \psi(x, v) + \alpha \frac{\partial \psi}{\partial v} \tag{1}$$

$$\psi = \psi(x, v), \qquad v \in R, \qquad x \in R^+$$

i.e., the one-dimensional linear stationary Fokker–Planck equation, which describes the steady one-dimensional Brownian motion of a classical particle of mass m in an isotropic fluid in thermal equilibrium at the temperature T, on which also acts an external conservative force.[11,12]

In Eq. (1) the range of the velocity is R, while the position x is restricted to the right half-axis; without any loss of generality we can take the "friction coefficient" $\gamma = 1$ and also $kT/m = 1$ by a suitable choice of the units. At the wall $x = 0$ we can impose the following boundary condition:

$$\psi(0, v) = \psi_0(v) \quad \text{(given)}, \qquad v > 0 \tag{2}$$

We shall restrict our discussion to the case $\alpha > 0$, because we cannot expect a regular behavior at ∞ when $\alpha < 0$, i.e., when the particles have an acceleration driving them toward infinity.

We remark that there are essentially two kinds of possible behaviors at ∞, according to whether there is a net flow of particles through the wall or not. Here we restrict our discussion to the case when the net flow vanishes at ∞. This is, e.g., the case of particles suspended in the atmosphere under the action of gravity.

In this case we expect an equilibrium solution of the form (Maxwell–Boltzmann distribution in a force field)

$$\psi_{\text{eq}}(x, v) = C \exp\left(-\frac{v^2}{2} - \alpha x\right) \tag{3}$$

This is the form of the solution that should prevail outside the kinetic layers.

It would be tempting to write $\psi(x, v) = \psi_{\text{eq}}(x, v)\,\phi(x, v)$ and look for $\phi(x, v)$. This, however, does not turn out to be the most convenient choice.

Let us introduce the new unknown $\phi(x, v)$ related to ψ by the following relation:

$$\psi(x, v) = \exp\left[-\frac{v^2}{4} - \frac{\alpha}{2}(x + v)\right] \phi(x, v) \tag{4}$$

The equation for ϕ now is

$$v\frac{\partial \phi}{\partial x} = \frac{\partial^2 \phi}{\partial v^2} - \left(\frac{v^2}{4} + \frac{\alpha^2}{4} - \frac{1}{2}\right)\phi(x, v) \tag{5}$$

which can be written in the following operational form:

$$T\frac{\partial \phi}{\partial x} = -A\phi \tag{6}$$

where

$$(Tf)(x, v) = vf(x, v) \tag{7}$$

and

$$(Af)(x, v) = -\frac{\partial^2 f}{\partial v_2} + \left(\frac{v^2 + \alpha^2}{4} - \frac{1}{2}\right) f(x, v) \tag{8}$$

Since we want ψ to behave as ψ_{eq} defined by Eq. (3) when $x \to \infty$, we have the following boundary conditions for ϕ:

$$\phi(0, v) = \phi_0(v) \quad \text{(given)}, \qquad v > 0 \tag{9}$$

$$\lim_{x \to \infty} \phi(x, v) = C \exp\left[-\frac{v^2}{4} - \frac{\alpha}{2}(x - v)\right] \qquad (C = \text{const}) \tag{10}$$

Let us introduce the following Hilbert spaces:

$$H = L^2(R; dv), \qquad H_T = L^2(R; |v|\, dv)$$

together with the scalar products and corresponding norms:

$$(u, v) = \int u(v)\, z(v)\, dv, \qquad \|u\| = (u, u)^{1/2} \tag{11}$$

$$(u, z)_T = \int u(v)\, z(v)\, |v|\, dv, \qquad \|u\|_T = (u, u)_T^{1/2} \tag{12}$$

Equation (5) together with the boundary conditions (9) is an indefinite Sturm–Liouville problem. We want to show that it can be solved by a suitable application of a theorem established by Beals[4] and extensively applied to this kind of problems. To this end, we have to introduce some additional notation. The theory developed by Beals that we recall briefly in the next section, and a systematic exposition of which is contained in ref. 10, allows one to deal with more general equations of the form

$$w(\mu) \frac{\partial \psi}{\delta x}(x, \mu) = \frac{\partial}{\partial \mu}\left[p(\mu) \frac{\partial \psi}{\partial \mu}(x, \mu)\right] - q(\mu)\, \psi(x, \mu) \tag{13}$$

[where μ ranges over an open subset I of the real line, and the real-valued function $w(\mu)$ (the weight) changes sign on I], endowed with forward–backward spatial boundary conditions in half-space geometry:

$$\psi(0, \mu) = \phi_+(\mu) \qquad \text{for those } \mu \text{ where } w(\mu) > 0 \tag{14a}$$

$$\|\psi(x, \mu)\| = O(1) \quad \text{or} \quad o(1) \qquad x \to \infty \tag{14b}$$

Let us introduce the orthogonal projections Q_+ and Q_- of H onto maximal T-positive and T-negative T-invariant subspaces:

$$(Q_\pm h)(\mu) = \begin{cases} h(\mu), & \mu \in I_\pm \\ 0, & \mu \in I_\pm \end{cases}$$

where $I_\pm = \{\mu \in I\colon \pm w(\mu) > 0\}$.

We can write (14a) in the following form:

$$Q_+ \psi(0) = \phi_+$$

3. PREVIOUS RESULTS ON INDEFINITE STURM–LIOUVILLE PROBLEMS

If we perform a formal separation of variables of the kind $\psi(x, \mu) = \exp(-\lambda x)\, y(\mu)$, we obtain the Sturm–Liouville boundary value problem:

$$-[(py')' - qy] = \lambda w y \tag{15}$$

with the same boundary conditions (self-adjoint). The theorem proved by Beals[4] requires the following preliminary hypotheses:

I. $I_{\mp} = \{\mu \in I \text{ such that } \pm w(\mu) > 0\}$ are nonempty finite unions of open intervals, and the set $I_0 = \{\mu \in I\colon w(\mu) = 0\}$ has finite cardinality.

II. The function $w(\mu)$ is continuous on $I_+ \cup I_-$.

III. In a neighborhood of each sign change $\mu_0 \in I$ of the weight function $w(\mu)$ there exists a C^1-function m satisfying $w(\mu) = \operatorname{sign}(\mu - \mu_0)(\mu - \mu_0)^\tau\, m(\mu)$ with $\tau > -1/2$ and $m(\mu) \neq 0$.

IV. The function $p\colon I \to R$ is locally absolutely continuous and strictly positive on I; the function $q\colon I \to R$ is continuous on I.

Furthermore, let us denote by D_1 the linear subspace of functions $h \in H$ that are absolutely continuous on I and whose derivatives h' satisfy $p\,|h'|^2 \in L^1(I, d\mu)$ and $q\,|h|^2 \in L^1(I, d\mu)$ and define the following sesquilinear form:

$$(h, g)_A = \int p(\mu)\, h'(\mu)\, \bar{g}'(\mu)\, d\mu + \int q(\mu)\, h(\mu)\, \bar{g}(\mu)\, d\mu \tag{16}$$

The existence of a linear subspace $D \subset D_1$ is assumed containing the compactly supported C^1-functions on I and of a finite-dimensional subspace $N_0 \subset D$ with the following properties:

V. $(wh, wh) \leqslant c(h, h)_A$ for some constant c and all $h \in D$ orthogonal to N_0 in H.

VI. $(h, h)_A \geqslant 0$ for all $h \in D$.

VII. $(h, h)_A = 0$ for all $h \in N_0$.

VIII. $\|h\|^2 \leqslant c(h, h)A$ for some constant c and all $h \in D$ orthogonal to N_0 in H.

The definition of the domain of the operator A requires some care. Let us define on D the inner product

$$(h, g)_1 = (h, g)_A + (h, g) \tag{17}$$

and denote by H_A the completion of D with respect to this inner product; since N_0 is of finite dimension and we assumed hypothesis VIII, H_A is continuously and densely imbedded in H. Let H'_A be the dual of H_A with respect to H. If we extend the inner product (17) to H_A, we may realize an operator A_0 from H_A into H'_A by $(A_0 h, g) = (h, g_A)$ for $h, g \in H_A$. We now define $D(A) = \{h \in H_A\colon A_0 h \in H\}$ with $A = A_0$ on $D(A)$.

We have also to require some compactness assumption, i.e., that the multiplication by $w(\mu)$ defines a continuous mapping from H_A to H; in the case in which A has compact resolvent, the inclusion $H \subset H'_A$ is compact and in this case compactness follows directly from the continuity of the map from H_A to H.[4,10]

It is also possible to prove that the operator $T^{-1}A$ with domain $D = \{h \in D(A)\colon Ah \in T[H_A]\}$ is closed with respect to the H_A-topology.[10] The main steps in obtaining Beals' theorem are an adaptation of a lemma proved by Baouendi and Grisvard,[15] already utilized by Beals and Protopopescu,[1] and a theorem which states the equivalence between the scalar products $(\circ, \circ)_s$ and $(\circ, \circ)_T$. We recall both the lemma and the theorem of equivalence:

Lemma 1. There exist bounded linear operators X and Y^* on H_A satisfying

$$XQ_+ = Q_+, \qquad |T|\, X = Y^*T$$

on H_A, with Y^* the adjoint of Y in H.

Theorem 2. The inner products $(\circ, \circ)_S$ and $(\circ, \circ)_T$ are equivalent on H_A and therefore $H_T \simeq H_S$.

We remark that a proof of this theorem alternative to that of Beals was given by Curgus,[14] who showed that the identification $H_T \simeq H_S$ is equivalent to infinity being a regular critical point of S in a suitable indefinite inner product on H_T. Finally we recall the fundamental theorem by Beals,[4] which is the main tool of this work:

Theorem 3. Under the assumptions listed above, there is a sequence of eigenfunctions $\{u_n\}$ with eigenvalues $\{\lambda_n\}$ which is a basis for $L^2(I_+ \cup I_-;\ |w(\mu)|\, d\mu)$. If $\dim\ker A = 0$, then $\{u_n^+;\ \lambda_n > 0\}$ is a basis for $L^2(I_+;\ w(\mu)\, d\mu)$ and $\{u_n^-;\ \lambda_n < 0\}$ is a basis for $L^2(I_-;\ w(\mu)\, d\mu)$.

In the same paper Beals showed also that if $\dim\ker A = \operatorname{span}\{u_0\}$ under the same assumptions as before, u_0 must be included with $\{u_n^+;\ \lambda_n > 0\}$ to obtain a basis for $L^2(I_+;\ w(\mu)\, d\mu)$ if and only if

$$\int w(\mu)\, u_0(\mu)\, d\mu \geqslant 0 \tag{18}$$

and this was an essential step in proving the completeness theorem for the FPE in the case without external force.

4. HALF-RANGE COMPLETENESS

In order to apply Beals' result on indefinite Sturm–Liouville problems, we recall the following, well-known result.

Lemma 4. The operator A defined by (8) is positive definite in $H = L^2(R; dv)$.

It is easy now to check that our problem is in the range of validity of Beals' theorem and that all the assumptions listed in the previous section are satisfied; all this can be done in strict analogy with the case $\alpha = 0$. In particular, it is straightforward to verify that multiplication by v (the weight function w in our case) defines a continuous mapping from H to H_A, while the compactness of the mapping from H_A to H'_A follows by the compactness of the resolvent of A.

Let us examine the spectrum of A in some detail; the eigenvalue equation for the operator A can be written in the form

$$\frac{\partial^2 g}{\partial v^2} + \left(\frac{1}{2} - \frac{v^2}{4} - \frac{\alpha^2}{4} + \frac{\lambda^2}{2}\right) g = 0 \tag{19}$$

This is essentially the well-known Weber equation,[2,3] whose solutions are the so-called parabolic cylinder functions. In correspondence with every eigenvalue

$$\lambda_n = \left(2n + \frac{\alpha^2}{2}\right)^{1/2} \tag{20}$$

there is an eigenfunction of the kind

$$u_n(v) = c_n D_n(2q_n - v) \tag{21}$$

with

$$q_n = (n - \alpha^2)^{1/2} \tag{22}$$

The role of these eigenfunctions in the eigenvalue problem $Au = \lambda Tu$ was pointed out by Marshall and Watson,[2] generalizing the result contained in a paper by Pagani[3] in the case where the term of the external body force is missing. It is immediate to see that the resolvent is compact and that the main difference from the case with $\alpha = 0$ is the absence of the null eigenvalue (the operator A is positive definite), which makes the problem is some respects easier to treat than the previous one. In conclusion, we can state the following result.

Theorem 5. The set of eigenfunctions u_n given by (28) is complete in the Hilbert space H and the solution of Eq. (1) with boundary conditions (2) and (3) can be expanded in series of such eigenfunctions.

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